University of California at Berkeley

Making Math Fun

A Specific example

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Introduction

This paper gives a proposal on what mathematics should be and how mathematics can be taught.

The focus is on the very young students and how to start them off on the right track. This will enable them to use mathematics more effectively in the later stages of their education. Math will then become a tool to examine the material more thoroughly with rather then a stumbling block as it often becomes.

In order to make the discussion clear I chose one important topic in mathematics and develop my thesis around it. It should be clear that this topic is just an example of how an advanced concept in mathematics can be broken down and delivered in steps.

The topic I chose as the example is an expansion of numbers called the Pascal Triangle. I chose this topic because there are already many books elaborating on how to break this expansion down and what properties the numbers and the expansion has. It can be broken down enough to show the concepts to young children, yet it is generally not brought into the discussion until late in the high school years. I do not spend much time discussing these properties, but will defer most of these discussions to an appropriate text[1].

[1] a good example of such a text is: "The Incredible Pascal's Triangle." Margaret J Kenney. Boston College Mathematics Institute, Motivated Math Project Activity. Boston College Press, 1976.

One of the things the that the Pascal Triangle does is help to determine the coefficients in a binomial expansion. For example:

$(x + y)^4 = 1*x^4 + 4*x^3y + 6*x^2y^2 + 4*xy^3 + 1*y^4$

where the set {1, 4, 6, 4, 1} is the fourth row of the Pascal Triangle.

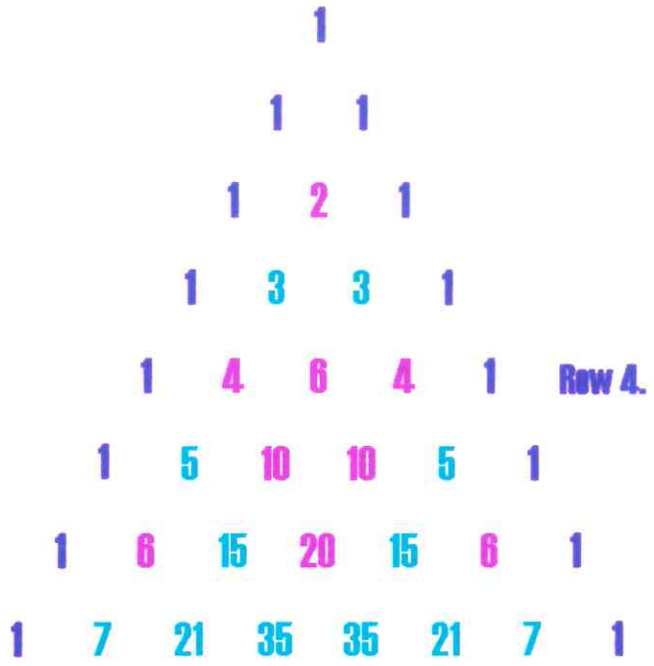


fig. 1: Numbering system

Background

The idea for the Pascal's tree came forth from a long series of discoveries, discussions and readings, not to mention over fifteen years of experience as a student. The Pascal's tree idea is in my view a fairly obvious example of a fun way of introducing some of the fundamental concepts in mathematics.

One of the co-authors to the paper "Elementary Math Curriculum", Sam Davis, did some research in the area of implementation of the type of math programs drawn up here. He found that there is much literature available on the subject. There is a wealth of creative proposals already in existence. All one needs is to type something like "math and games" or "math

and curriculum and projects and elementary school" on an ERIC[2] terminal and it becomes obvious that there has already been much work done in this area[3]. This means that if a teacher or a school wanted to, they could easily get plenty of information to start immediately!

There is of course an almost infinite amount of ground left to be covered so there is plenty of room for the creative input of educators everywhere. There are literally hundreds of topics that are now not usually introduced until late high school or college that could easily be introduced at the early stages of education. This can be done without straining the educational system or overloading the students. On the contrary, fundamental changes in how math is taught to students could lighten the load while vastly increasing the amount learned simply by making the subject matter more interesting.

It should become clear from the example I present in this paper that the introduction of these topics does not necessarily mean that large investments need to be made on expensive equipment. There are of course clear advantages to having tools such as computers available, but one does not need to wait for that. Some paper, a pencil, and a bit of creativity is more than adequate to get started. All that is really required is that the creative ideas be brought to educators and that they be familiar with the subject matter.

A general lack of familiarity on the part of the instructors with this material could be a large hurdle since these programs do require a large amount of background knowledge to be presented properly. Many of the educators came through an educational system that itself stressed arithmetic and these topics may seem strange and inaccessible to them at first. The hurdle needs to be overcome, however.

Proper teacher training and literature needs to become widely available and the fundamental idea itself needs to be accepted. It is not enough for it to be set forth by the California State Department of Education[4], it needs to be generally accepted by the teaching community. The fundamental idea that advanced

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[2]ERIC stands for Educational Research Index Catalogue. It is available in many libraries as well as at some internet sights.

[3]A great example of an exploration into this area is the book by Lynn Arther Steen, On the Shoulders of Giants, New Approaches to Numeracy. National Press, 1990.

[4]California State Department of Education, Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve. 1985

forms of mathematics are at once interesting, useful, and easy to teach (given proper training) is fundamental to the advancement of science education. In fact the level of abstractness is one of the primary areas where math education in countries such as The Netherlands excel[5] over ours here in the United States.

The main thing that needs to be done is to make the programs readily available to teachers while giving them access to the tools they require to implement the projects. The tools need to be clearly outlined and simple to use so that current educators can easily apply them without having to do a lot of research themselves. This means that there is a need for further thought into math education, but that the primary step that needs to be taken is to make the material more accessible. New math programs need to be carefully thought out and presented in a very simple to understand way using clear diagrams and outlines. They then need to be brought to the teachers so that they do not need to go looking for the material in university libraries[6].

On the positive side, there have been countless books written on the topic of how boring arithmetic is. Many of them outline ideas on what can be done to improve math education. As I discovered while doing the group project that this paper stems from, there are many projects of all sorts being attempted. The necessity to explore and implement such ideas have even been written in the California curriculum outline, Framework 1985[7].

We could very well be entering a new era in mathematics education. The tools, both simple creative ones and complicated expensive ones, are becoming readily available. Hopefully the current debates flared up by the campaign to bankrupt the already strained public school system will encourage educators and school boards to be creative and reach out to new and better methods of education.

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[5]Opgaven wiskunde havo, M. Kindt, H. Steur, Nederlands Vereniging van Wiskunde. Wolters-Noordhoff bv Groningen. 1975. This is an outline of the general mathematics curriculum for college bound student in The Netherlands.

[6]This opinion is actually a paraphrase of a conversation I had with Ms. Roberta Buchanan. Ms. Buchanan has a masters in education and has been teaching at elementary schools for 25 years.

[7]Framework, op. cit.

Example Project

Pascal Triangle

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In this paper I examine one example for a creative far reaching mathematics project. One of the many powerful mathematical tools that can easily and creatively be introduced to very young children is the Pascal triangle.

The Pascal triangle was named after Blaise Pascal (1623-1662). Blaise Pascal, a 17th century mathematician, who wrote about some of the properties of the triangle in his book "Traité du Triangle Arithmétique". The triangle, however, can be traced back as far as 1100 A.D. to China and Persia[8]. It has been a topic of interest in mathematics for many years!

The triangle has found its way into many branches of mathematics. It has an incredible number of interesting properties that are not just simply mathematical curiosities, but have real value as mathematical tools.

When to teach

The concepts encompassed by the Pascal Triangle can (and should) be developed over many years starting when children are very young. The concepts can be implanted into the young students mind so that years later, when a particular topic is developed, the basic concepts will already be available to the students. This will allow the students to develop far greater insight into the topic. If the students are already familiar with the concepts of the triangle as well as a host of other equally important topics, the basic background will be far better understood and the subject will not only be easier for the students to grasp, they will be capable of a far greater understanding. The topic will be much more interesting to them.

The triangle can be introduced as early as first grade. It can then be re-examined almost on a yearly basis to the children as the children develop more skills that can be applied to the

[8]The Incredible Pascal's Triangle. Margaret J Kenney. Boston College Mathematics Institute, Motivated Math Project Activity. Boston College Press, 1976.

triangle. Advanced high school students who had been introduced to the triangle and its related numbers and have re-examined it on a regular basis, will have distinct advantages in algebra, geometry, trigonometry, binomial functions, differential calculus, series functions, number theory, statistics, and the list goes on. If a student continues on to take courses in physics, electronics, computer science, economics, material science, quantitative social sciences, etc. etc., binomials come into play.

Thoroughly understanding binomials along with the related thought processes that stem forth from the triangle project will be a great asset throughout a students education and beyond. As complex algorithms and complicated systems become more commonplace it becomes ever more important that an overall understanding of the background theories underlying the foundations of these systems be thoroughly understood early on. This will allow the students to advance more rapidly while learning the more encompassing subject at hand, which is the ultimate reason the students needed to learn mathematics in the first place. During the later stages of education, math should be available as a tool to help understand concepts much the way other languages are; it should not be a stumbling block.

Finally, the triangle is a good hands on diversion. As computer 'games' become more and more a part of education, already dominated by watching and listening, a physical triangle, made from cardboard, implementing ideas, should be an interesting, fun experience. And it is inexpensive too!

Uses

When introducing the Pascal Triangle to very young children it should be done in a playful manor. It should be an interdisciplinary activity combining math with art and play. It can be used as a tool to teach children how to measure geometric shapes and judging proportionality. It will give the children valuable insight into structure along with some early familiarity to the triangle while developing their motor skills. It can also be a very powerful group project, teaching students to work together, cooperate and follow instructions.

For the teacher it is easy to see how the students are progressing since the geometric properties will make errors very noticeable and clearly correctable.

There are many levels of learning tied into this project, so it can be used over and over again by a group of children for many years.

In parallel with the project itself the teacher can also take advantage of the almost limitless number of thought games that have been developed to go along with this particular triangle and its related numbers. The thought games range from simple counting and spatial exercises, to complicated puzzles, and theoretical questions. An instructor can use these games as tools that allow gifted students to discover mathematical theories with little more then a few suggestions.

A Pascal's Tree

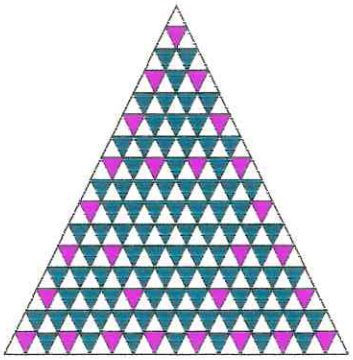


fig. 2: The Pascal's Tree

One way to start introducing the Pascal triangle to very young children is as a Pascal's tree. It can be a fun alternative to the standard decorative tree for a first grade class.

The children, with the teacher's help, can make a large triangle and **systematically** divide it up into small unit triangles. The unit triangles will later be filled with numbers to complete the triangle.

Interesting patterns will develop if the triangle is set up properly. The tree should be made out of decorative colors patterned in a strategic way. Later on the decorative patterns will begin to develop meaning. The self symmetry and the numerical significance of the patterns should be left as a surprise for the students to discover at their own pace. The discoveries should be left to the individual student's own insight in the hope that the discovery will spark interest that could lead to self-motivated further investigation.

It is important for the overall learning experience that the children be allowed to do almost all the work on building the triangle themselves. It does not matter if the triangle does not work out on the first attempt. The construction of the triangle

should be viewed as an art project. If it does not turn out as planned, then it can still serve as a decorative tree and the children should receive praise for the tree in and of itself. The value of the project will not be lost by the creation of a variant. The tree can be discussed in class no matter what form it takes. One could for example have the children count the triangles, or estimate how much of the tree is green. Counting the triangles could be set up as a game where the children are given a few days to find as many of them as they can, then at the end of the week the class as a whole could point out the triangles while someone keeps track of the number of triangles on the board. Estimating area should probably be done as a class discussion led by an instructor.

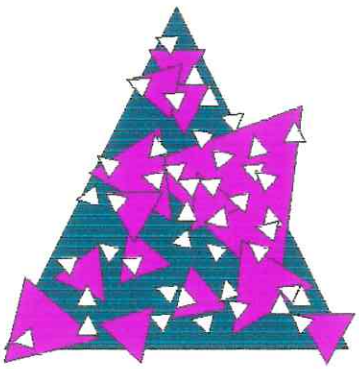


fig. 3: Random Triangles

If the class made several trees, all the trees should be brought into the discussion with equal enthusiasm. Try to find some unique characteristic in each tree that can be used in the discussion. It is very important that no tree be singled out as the best. The idea is to pull the entire class into the discussion and singling out one tree as the best is sure to alienate a large portion of the class. The true beauty of the tree is that it can be brought back next year. The children can start again with a new set of instructions. Each year the children can be given more and more of the initiative when constructing the tree. By the third or fourth year the teacher should be able to simply tell the children to draw a Pascal's tree. By this point the children should be fascinated by what new secrets they will learn about it.

The symmetry of the triangle has many advantages. It can be used to break the triangle up into pieces so that smaller groups can work on sections of the tree. It will allow an instructor to quickly see how the tree is working out making the project easy to oversee. It also serves as an aid for when the children start putting numbers in the tree. One mistake will quickly propagate

and cause the overall pattern of numbers to be off. Also, the axis of symmetry will allow the instructor to compare results on the two halves of the triangle. The children should not be told about the symmetry, they should be allowed to discover it on their own.

The Pascal's tree can be saved so that the children can refer to it in later years. The tree can be used to teach children how to add small and large numbers, measure, build, count, multiply, divide, play counting games, solve puzzles, and the list goes on. It can be used by the teacher to teach algebra, geometry, fractions, etc. etc. With just a little creativity the possibilities are limitless. There are many books written on things that can be done with the triangle. A good starting book is Margaret Kenney's book, "the Incredible Pascal Triangle"[9]. Just let them have fun with it!

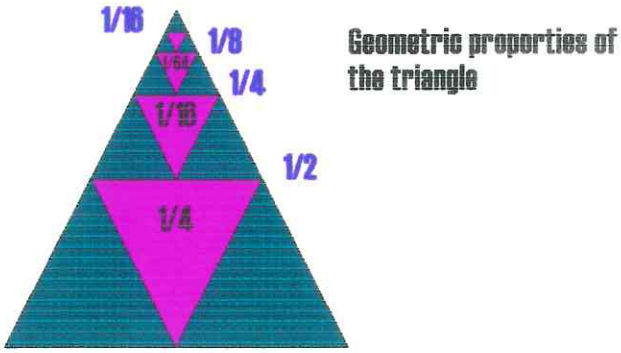


fig. 4: Geometric Proportions

Making a tree

Making a Pascal's tree is a simple inexpensive process and does not need to take up a lot of class time. The size of the project can be varied, but a 31 row[10] triangle is probably a good start.

Start with a rectangular piece of cardboard that is either 160cm x 80cm or 6ft x 8ft on a side, see drawing below. Either the metric or the English measuring systems can be used for the project, the dimensions work out nicely in both systems and they both have different advantages. I much prefer the metric system, however, since it leads to much less confusing mathematics when examining the areas of the triangles, or when using things like the Pythagorean theorem, etc.

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[9]The Incredible Pascal's Triangle. op. cit.

[10]Note that the isolated "1" at the top of the triangle is row zero, so a 31 row triangle actually has 32 steps. The steps are numbered: 0, 1, 2, 3, 4, ...

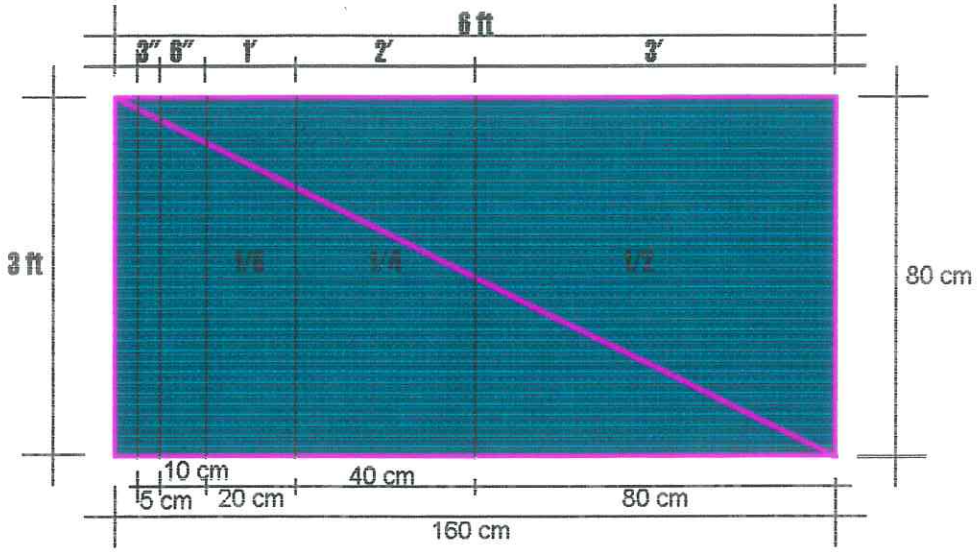


fig. 5: Dimensions of the tree

Cut the rectangle diagonally and flip one half over to form an equilateral triangle. Tape the triangle together and paint it the color the kids want the tree to be. I chose green. This process is then continued using rectangles that are 1/4 the area of the previous rectangle. All these rectangles can be painted a second color. I chose red in the drawings.

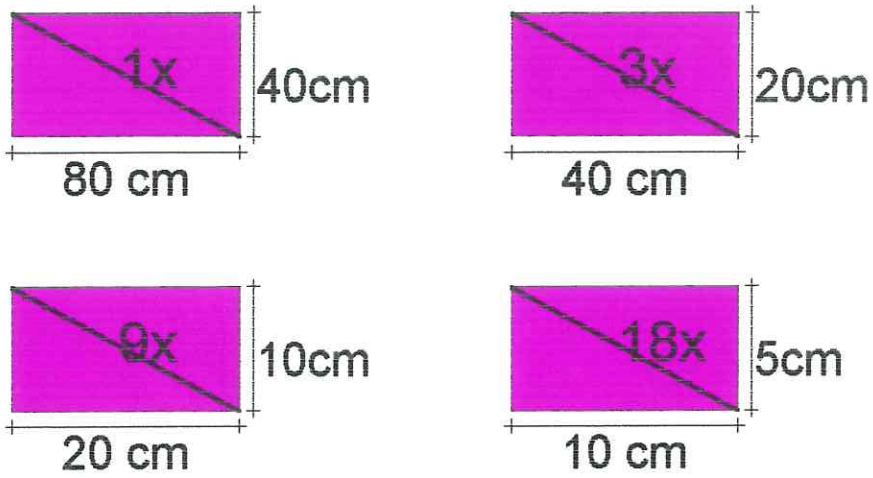
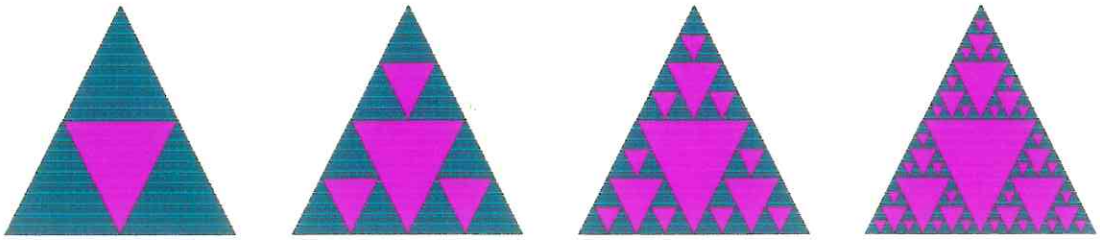


fig. 6: Dimensions of the triangles

Finally, the entire tree should be covered with the smallest set of triangles, using the larger triangles as guides. As a fun alternative the small triangles could be taped only along the base so they can hinge up and down. This would allow the triangle to be used in some adding games, or to hide some sets of numbers while others are being examined.



Recursively adding triangles with 1/4 the area.

fig. 7: Self Symmetry

Numbering the tree

I will not discuss the theories or peculiarities of the triangle here, there are many books on that subject including Pascal's own book "Traité du Triangle Arithmétique" published in 1665. I will, however, briefly explain how to obtain the numbers just for the sake of completeness.

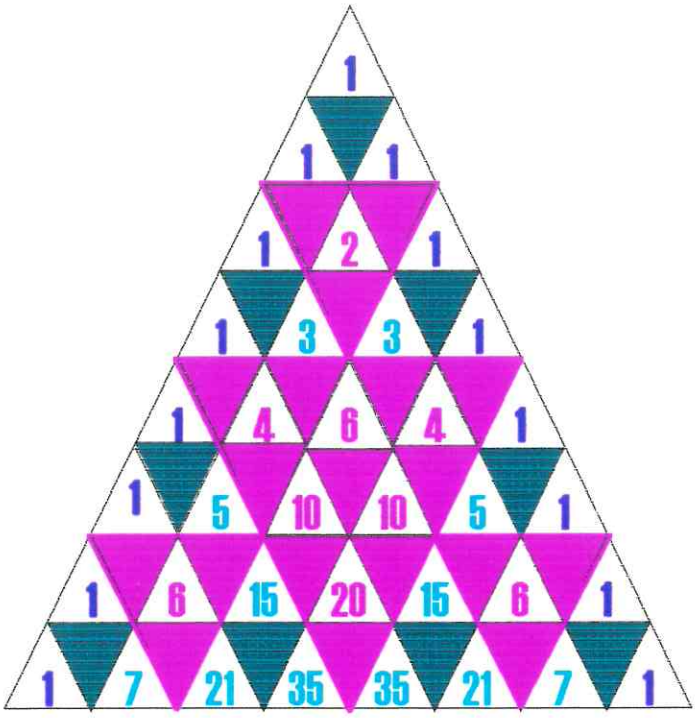


fig. 8: A numbered tree

Starting with the triangle oriented like a tree put 1's in all the small outermost triangles that share the same orientation as the tree itself. Then, starting at the upper empty triangle, place the

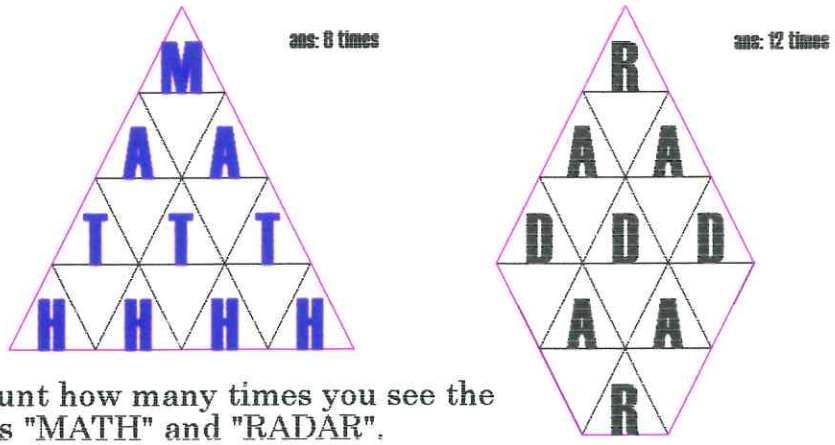
sum of the two numbers immediately above it in the triangle. This sum will be "1+1=2". Continue this process, always filling a triangle with the sum of the two numbers that are immediately above it. For example the next row down will be {1, 3, 3, 1}. Refer to diagram above.

The numbers will rapidly become quite large. In fact the largest number in a triangle with 31 rows will be 300,540,195[11]. This is a very large number indeed, but it is not so large that it is tiresome to write. Plus, I remember when I was a kid, I loved large numbers!

More Fun

Once the triangle is constructed you can keep it on the wall as decoration as well as occasionally bringing it into the lecture discussions. You can also set it up as a game that you allow students to play when they have some extra time, or as something they can take home and play with instead of watching TV. Children can even build up their egos by stumping their parents with puzzles that they can figure out just by virtue of guidance and persistence.

Here are a few examples. There are many more waiting in the library-if they are not there, tell your librarian, there should be!



I. Count how many times you see the words "MATH" and "RADAR".

fig. 9: Word games

II. Find the rows with all odd numbers in the tree?

III. What would the next row be that has only odd numbers?

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[11]This number was obtained using binomial theorem. It occurs twice at the very center of the base, just as one might expect.

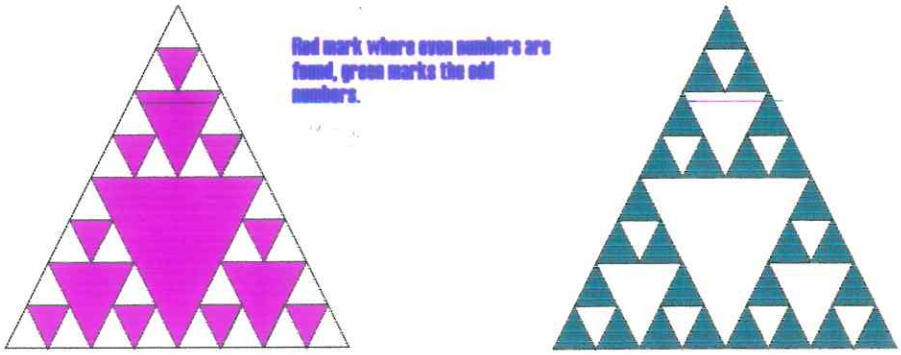


fig. 10: Locating even and odd numbers

IV. Find all the even numbers?

V. How many numbers are in row x?

VI. How many numbers are above row x?

VII. What is the area of a triangle compared to the area of the tree?

The list of questions goes on. Think of some questions yourself. Even better, encourage the children to think of questions. You could even give a prize for the best question. The idea is to get the children to learn without feeling like they are working. Most importantly, have fun with it.

Conclusion

The Pascal's tree is an excellent example of how advanced ideas in mathematics can be brought to young children. There are many other topics that can be presented in such an abstract way.

There needs to be a break away from the old status quo of teaching arithmetic to young children. Math should be presented as a fun topic and never used as punishment! There are many wonderful pilot programs being attempted and there is a vast amount of available research.[12] There needs to be a wider acceptance and general implementation of such programs however. This paper gives a clear example of what can be done.

There will be a need for training and some new materials for the teachers, but the cost of that can be minimized by bringing in individual projects such as the Pascal's tree project. As more and more projects make their way into the educational system as staple projects, the curriculum will begin to change itself.

The later part of the K-12 system is a little more difficult to deal with however. Follow up on these projects is crucial in having them give the highest value to the students. A student needs time to develop the skills and then needs to be led to discovery. That is where the real challenge lies. High school mathematics will eventually need to be transformed into a much broader, far more abstract topic, embracing art and culture as well as simple arithmetic and calculus. But it all starts in elementary school. Giving children a good early start will make the job much easier for junior high school and high school teachers. This will be specially true if tools such as computers are available for motivated students to pursue their interests on their own initiative.

In this area there is actually a large amount of progress being made. There are many wonderful programs being developed taking advantage of the rapid advancements in multi-media technologies. This means students will be able to benefit greatly from projects that encourage self motivated thinking. If the motivated students are able to teach themselves in such a way,

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[12]For a discussion of this see the paper "Elementary Mathematics Curriculum", EDU 190 Group Project.

schools will be able to free up more of their valuable human resources to work with the less motivated students.

There are plenty of reasons to make math fun. This paper gives an example of how to both make math fun while greatly expanding the depth of what is being learned by the students. If you are interested in the Pascal's tree project, my suggestion is to make a small one of your own and play with it for a bit and let the children get started. It is a very simple project and its costs are insignificant. If one is dealing with very young children one may want to pre-cut the various triangles as laid out in "making a tree" and just let them make a random tree. One could, for example, give them the bigger triangles first, followed by the smaller ones. Older children may be able to make a tree given only written, or verbal, instructions. Remember, all trees are worthy of praise.

It might help you to visit a local library for a book with games that you can use to challenge the class with now and again. The Pascal triangle is truly unlimited and it is an excellent example of how math should be taught.